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Rates of convergence of multipoint rational approximants and quadrature formulas on the unit circle[1]

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Abstract

In this paper, multipoint rational approximants to the Riesz–Herglotz transform of a Borel measure μ, supported on $[-\pi,\pi]$ are considered. We give estimates for the rate of convergence of these approximants. These estimates are obtained from the asymptotic behaviour of a sequence of rational functions, orthogonal with respect to μ and having prescribed poles $1/\overline{\alpha}_k$, where $\{\alpha_k\}_1^\infty$ is a sequence of complex numbers compactly contained in the open unit disk. As an application we also give the rate of convergence of rational Szegő quadrature formulas for integrals with respect to μ.

Keywords: Multipoint rational approximation; Orthogonal rational functions; Quadrature formula

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1. Introduction

For a given positive Borel measure μ supported on the interval $[-\pi,\pi]$, the Riesz–Herglotz transform is defined as [1]

$$F_\mu(z)=\int_{-\pi}^{\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\,d\mu(\theta)=\int_{-\pi}^{\pi}D(t,z)\,d\mu(\theta),\quad D(t,z)=\frac{t-z}{t+z},\quad t=e^{i\theta}. \tag{1.1}$$

Throughout the paper, we use the notation $\mathbb{C}$ for the complex plane, $\hat{\mathbb{C}}=\mathbb{C}\cup\{\infty\}$ for the extended complex plane and $\mathbb{T}=\{z\in\mathbb{C}:|z|=1\}$, $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, and $\mathbb{E}=\{z\in\mathbb{C}:|z|>1\}$, for the unit circle, the unit disk and the exterior of the unit disk, respectively.

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We are mainly concerned with the study of sequences of modified rational approximants interpolating F_μ at the points $\{\alpha_k\}_0^\infty \subset \mathbb{D}$ and at the points $\{1/\overline{\alpha}_k\}_0^\infty \subset \mathbb{E}$. The precise definition of these approximants is explained in Section 3. Assume that $R_n(z)$ is such an approximant and $E_n(z) = F_\mu(z) - R_n(z)$ the corresponding error. Section 3 contains integral expressions for $E_n(z)$. The rate of convergence of the errors is investigated in Section 5. More precisely, it is investigated under what conditions the sequence $\{E_n(z)\}$ satisfies

$$\limsup_{n\to\infty} |E_n(z)|^{1/n} \leqslant h(z) < 1, \tag{1.2}$$

possibly for any $z \in \hat{\mathbb{C}} - \mathbb{T}$.

In previous papers [3, 5, 10], the authors studied orthogonal rational functions with poles among the prescribed points $\{1/\overline{\alpha}_k\}_1^\infty$ and orthogonal with respect to μ. These play a fundamental role also in this paper and therefore we shall recall the relevant properties of these functions in Section 2. These are used in Section 3 to derive the integral expressions for the interpolation error $E_n(z)$. From the formula (1.2) it turns out that to settle the problem for the rate of convergence, we should know the root asymptotic behaviour of the orthogonal rational functions. Such results are obtained in Section 4 and applied in Section 5. The modified approximants can be used to construct the so-called rational Szegő formulas to approximate integrals of the form

$$I_\mu\{f\} = \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\mu(\theta).$$

These quadrature formulas are extensions of the quadrature formulas introduced by Jones et al. [19] in connection with the solution of the trigonometric moment problem. From the estimates for the rate of convergence for the multipoint rational approximants, we therefore easily derive estimates for the rate of convergence of these quadrature formulas. Some preliminary results for the convergence of multipoint rational approximants and these rational Szegő formulas were obtained in [11].

2. Preliminaries

Orthogonal rational functions will play a fundamental role. We introduce their definition and their main properties.

Let $\alpha = \{\alpha_k\}_0^\infty$ be a given sequence in $\mathbb{D}$ with α_0 fixed to be 0 and consider the nested spaces $\mathscr{L}_n$ of rational functions of type (n,n) (i.e., their numerator and denominator have degree n at most) which are spanned by the basis of Blaschke products $\{B_k\}_0^\infty$, where

$$B_0 = 1, \quad B_n(z) = \zeta_1(z)\cdots\zeta_n(z), \quad n = 1,2,\ldots \quad \text{with } \zeta_k(z) = \frac{\overline{\alpha}_k}{|\alpha_k|}\frac{\alpha_n - z}{1 - \overline{\alpha}_n z}.$$

By construction we set $\overline{\alpha}_k/|\alpha_k| = -1$ for $\alpha_k = 0$. Thus $B_k(z) = z^k$ when all $\alpha_k = 0$, thus in that case $\mathscr{L}_n = \Pi_n$, the space of polynomials of degree at most n. Introducing

$$\omega_n(z) = \prod_{j=1}^{n}(z - \alpha_j) \quad \text{and} \quad \pi_n(z) = \prod_{j=1}^{n}(1 - \overline{\alpha}_j z), \tag{2.1}$$

we can set

$$B_n(z) = \frac{\omega_n(z)}{\pi_n(z)}\eta_n, \quad \eta_n = (-1)^n \prod_{j=1}^{n} \frac{\overline{\alpha}_j}{|\alpha_j|}$$

and

$$\mathscr{L}_n = \text{span}\{B_k : k = 0, 1, \ldots\} = \left\{ \frac{p(z)}{\pi_n(z)} : p \in \Pi_n \right\}.$$

Thus $\mathscr{L}_n$ is the space of rational functions with poles among the prescribed points $\{1/\overline{\alpha}_k\}_1^n$.

Given the positive measure μ, an inner product is defined as

$$\langle f, g \rangle = \langle f, g \rangle_\mu = \int_{-\pi}^{\pi} f(e^{i\theta})\overline{g(e^{i\theta})}\, d\mu(\theta) = I_\mu\{f\overline{g}\}.$$

Orthonormalizing the basis functions $B_0, B_1, \ldots$ with respect to this inner product gives a sequence of orthonormal rational functions $\{\phi_k\}_0^\infty$. We assume that μ is normalized by $\int_{-\pi}^{\pi} d\mu(\theta) = I_\mu\{1\} = 1$, so that, for example, $\phi_0 = 1$. Furthermore, $\phi_k \in \mathscr{L}_k - \mathscr{L}_{k-1}$ and $\phi_k \perp \mathscr{L}_{k-1}$, $k = 1, 2, \ldots$.

We also introduce the substar conjugate transformation $f_*(z) = \overline{f(\hat{z})}$, $\hat{z} = 1/\overline{z}$ which allows to define for $f_n \in \mathscr{L}_n - \mathscr{L}_{n-1}$ the superstar conjugate as $f_n^*(z) = B_n(z)f_*(z)$.

In [3], it was proved that all the zeros of $\phi_n(z)$ are in $\mathbb{D}$, or equivalently, all the zeros of $\phi_n^*(z)$ are in $\mathbb{E}$. Moreover, the system $\{\phi_n^*\}_0^\infty$ satisfies the orthogonality properties

$$\langle \phi_n^* \rangle f = 0, \quad \forall f \in \mathscr{L}_n(\alpha_n) = \{f \in \mathscr{L}_n : f(\alpha_n) = 0\}.$$

Some other notation: $\mathscr{L}_{n*} = \{f_* : f \in \mathscr{L}_n\}$ and $\mathscr{L} = \bigcup_{n=0}^\infty \mathscr{L}_n$ and $\mathscr{L}_* = \bigcup_{n=0}^\infty \mathscr{L}_{n*}$ and for nonnegative integers p and q we set

$$\mathscr{L}_{p,q} = \mathscr{L}_{p*} + \mathscr{L}_q, \qquad \mathscr{R}_n = \mathscr{L}_{n,n} = \mathscr{L}_{n*} + \mathscr{L}_n = \mathscr{L}_n \cdot \mathscr{L}_n.$$

Note that when all $\alpha_k = 0$, then $\mathscr{L} = \Pi$ is the space of polynomials, $\mathscr{L}_{p,q} = \Lambda_{-p,q}$, the space of Laurent polynomials of the form $\sum_{j=-p}^{q} c_j z^j$, $c_j \in \mathbb{C}$. The space $\mathscr{R} = \bigcup_{n=0}^\infty \mathscr{R}_n$ is the rational generalisation of the space Λ of Laurent polynomials and in this paper, it will take the role played by Λ in [19, 8] (see also [17]). The so-called functions of the second kind, associated with ϕ_n were also introduced in [3] as

$$\psi_0 = 1, \quad \psi_n(z) = I_\mu\{D(t,z)[\phi_n(z) - \phi_n(t)]\}, \quad n = 1, 2, \ldots .$$

In fact, it can be shown that

$$\psi_n(z) = I_\mu\left\{D(t,z)\left[\phi_n(z) - \frac{f(t)}{f(z)}\phi_n(t)\right]\right\}, \quad \forall f \in \mathscr{L}_{(n-1)*}, f \not\equiv 0. \tag{2.2}$$

By taking the superstar conjugate we obtain

$$\psi_n^*(z) = I_\mu\left\{D(t,z)\left[\frac{f(t)}{f(z)}\phi_n^*(t) - \phi_n^*(z)\right]\right\}, \quad \forall f \in \mathscr{L}_{n*}(1/\overline{\alpha}_n), f \not\equiv 0, \tag{2.3}$$

where $\mathscr{L}_{n*}(w) = \{f \in \mathscr{L}_{n*} : f(w) = 0\}$. For the computation of $I_\mu\{f\}$, we shall introduce quadrature formulas of the form

$$I_n\{f\} = \sum_{j=1}^{n} A_{j,n} f(x_{j,n}) \tag{2.4}$$

such that $I_\mu\{f\} = I_n\{f\}$ for all $f \in \mathscr{L}_{p,q}$ with p and q as large as possible. It is well known that for quadrature formulas on an interval, one can take the nodes $x_{j,n}$ to be zeros of orthogonal polynomials, or in the rational case, of the orthogonal rational functions. In the present case where integration is over the unit circle, the zeros of the orthogonal rational functions are in $\mathbb{D}$ and are therefore not appropriate nodes, since we want the nodes $x_{j,n}$ to be on $\mathbb{T}$. Therefore, the functions

$$Q_n(z,\tau) = \phi_n(z) + \tau\phi_n^*(z), \quad \tau \in \mathbb{T}$$

are introduced. It is shown in [3] that the zeros of $Q_n(z,\tau)$ are all simple and lie on $\mathbb{T}$. These functions are called paraorthogonal; they are orthogonal to all functions in $\mathscr{L}_{n-1} \cap \mathscr{L}_n(\alpha_n)$. For the quadrature formulas, based on the zeros of $Q_n(z,\tau)$, we recall the following results proved in [4]; see also [9, 12].

Theorem 2.1. (i) *For each $n \geqslant 1$ there cannot exist an n-point quadrature formula of the form* (2.4) *with distinct nodes on $\mathbb{T}$ which is exact (i.e., $I_\mu\{f\} = I_n\{f\}$) in $\mathscr{L}_{n-1,n}$ or $\mathscr{L}_{n,n-1}$.*

(ii) *Let $x_{1,n}, \ldots, x_{n,n}$ be the zeros of the paraorthogonal function $Q_n(z,\tau)$, $|\tau| = 1$. Then there exist positive numbers $A_{1,n}, \ldots, A_{n,n}$ such that formule* (2.4) *is exact in $\mathscr{R}_{n-1} = \mathscr{L}_{n-1,n-1}$.*

Theorem 2.2. *Consider an n-point formula as in* (2.4) *with distinct nodes $x_{j,n} \in \mathbb{T}$. Then $I_n\{f\}$ is exact in $\mathscr{R}_{n-1} = \mathscr{L}_{n-1,n-1}$ if and only if the following two conditions are satisfied*

(i) *$I_n\{f\}$ is exact in $\mathscr{L}_{p,q}$, p and q nonnegative integers such that $p+q = n-1$.*

(ii) *If we write $\chi_n(z) = N_n(z)/\pi_n(z) \in \mathscr{L}_n$ with $N_n(z) = \prod_{j=1}^{n}(z - x_{j,n})$, then there exist complex numbers $\lambda_n \neq 0$ and $\tau_n \in \mathbb{T}$ such that $\chi_n(z) = \lambda_n[\phi_n(z) + \tau_n\phi_n^*(z)]$, i.e., χ_n is paraorthogonal.*

Thus, we see that a one-parameter family of quadrature formulas of the form (2.4) can be constructed where paraorthogonal rational functions play the role of the orthogonal polynomials or functions in the construction of Gaussian quadrature formulas. Quadrature formulas based on orthogonal polynomials or rational functions have recently been studied in [25, 16] for evaluation of integrals over $[-1,1]$.

Finally, in [9], the following result was proved.

Theorem 2.3. *Let $I_n\{f\}$, $n = 1,2,\ldots$ be a sequence of quadrature formulas characterized in Theorem* 2.2. *Then, if $\sum_{k=1}^{\infty}(1 - |\alpha_k|) = \infty$,*

$$\lim_{n\to\infty} I_n\{f\} = I_\mu\{f\}, \quad \forall f \in R_\mu(\mathbb{T}),$$

where $R_\mu(\mathbb{T})$ is the class of the integrable functions, i.e., for which $I_\mu\{f\}$ exists.

The quadrature formula $I_n\{f\}$ appearing in Theorem 2.2 will be called an n-point rational Szegő quadrature formula or an R-Szegő quadrature for short.

Remarks. 1. In Theorem 2.3, nothing is said about error bounds or rate of convergence. This is done in Sections 3 and 5 of this paper.

2. Initially, the parameters $\tau_n \in \mathbb{T}$ do not seem to have influence on the convergence of the resulting R-Szegő formulas. However, for the polynomial situation (i.e., $\alpha_k = 0$ for all k), some numerical experiments were reported in [17] (see also [26]). These experiments reveal that the ultimate accuracy reached can depend on the chosen value of τ_n. As far as we know, such influence has not been analysed theoretically, not even in the simple polynomial case.

3. Multipoint rational approximation

In this section and the subsequent ones, we consider rational functions of type (n,n) and, if there is no confusion possible, we shall not explicitly mention this fact.

It is well known that rational functions (of any type) play an important role in approximation theory, both from a theoretical and an applied point of view. This is clearly illustrated by classical Padé approximants (PA) which approximate a function maximally in one point, usually the origin or infinity. The function is assumed to be given in terms of the coefficients of its Taylor series or its asymptotic expansion at that point. Multipoint Padé approximants (MPA) are one of the possible generalizations where it is assumed that information about the function is known in more than one point. Although more general definitions can be given [14], for our purposes, it will be sufficient to start from a function f analytic in a (bounded or unbounded) region G of $\mathbb{C}$. It is assumed that we have information about f in a number of interpolation points in G. It is possible to consider the general situation where a non-Newtonian triangular array of interpolation points $\{\alpha_{j,n} : j = 1,\ldots,n; n = 1,2,\ldots\}$ is given but we shall not do this here. Because of the special character of the function $f(z) = F_\mu(z)$, it will be convenient for our purposes to consider not one but two sequences of interpolation points: $\alpha = \{\alpha_k\}_0^\infty$ and $\beta = \{\beta_k\}_0^\infty$ in G. Set

$$\tilde{\omega}_k(z) = \prod_{j=0}^{k}(z-\alpha_k) \quad \text{and} \quad \tilde{\pi}_k(z) = \prod_{j=0}^{k}(z-\beta_k), \quad k = 0,1,\ldots .$$

Let p,q and n be nonnegative integers with n fixing the type (n,n) of the rational functions considered. A rational function $F_n(z) = P_n(z)/Q_n(z)$ is said to be a multipoint rational approximant (MRA) to $f(z)$ of order $(p+1,q+1)$ in the weak sense if

$$[\tilde{\omega}_p(z)\tilde{\pi}_q(z)]^{-1}[f(z)Q_n(z) - P_n(z)] \tag{3.1}$$

is analytic in G and it is a MRA in the strong sense if

$$[\tilde{\omega}_p(z)\tilde{\pi}_q(z)]^{-1}[f(z) - P_n(z)/Q_n(z)] \tag{3.2}$$

is analytic in G. Since we shall allow an interpolation point to be ∞, we should be careful to interpret this definition for that special case. The rule is that in $\tilde{\omega}_n$ and $\tilde{\pi}_n$, the factors corresponding to ∞ are replaced by 1 [27] and if $l = q+1-\deg(\tilde{\pi}_q) > 0$, then (3.1) should be replaced by

$$f(z)Q_n(z) - P_n(z) = [\tilde{\omega}_p(z)\tilde{\pi}_q(z)]h(z)$$

with $h(z)$ analytic in G and satisfying $h(z) = \mathrm{O}((z^{-1})^{p+q+2})$ as $z \to \infty$. Similarly, if $l > 0$, we should add to (3.2) l extra interpolation conditions at ∞ given by

$$f(z) - P_n(z)/Q_n(z) = \mathrm{O}((z^{-1})^{l+1}), \quad z \to \infty.$$

Since the rational approximant P_n/Q_n depends on $2n+1$ parameters, the highest reachable order corresponds to $p+q = 2n-1$. The MRA with this maximal order are called multipoint Padé approximants (MPA). The MPA in the weak sense always exists. However (unlike the polynomial case where approximants of type $(n, 0)$ are considered), the existence of MPA in the strong sense cannot, in general, be guaranteed [24].

When $p+q = n-1$, we can fix the polynomial Q_n and it can be verified that a unique polynomial P_n exists so that (3.2) is satisfied. Such an approximant is called a multipoint Padé-type approximant (MPTA). More generally, one could fix a suitable part of the numerator and of the denominator. We then get partial Padé approximants (see [2, 15]). These approximants are however out of the scope of this paper.

The MPTA, i.e., the situation where $p+q = n-1$ and Q_n is a fixed polynomial is what Walsh calls rational interpolation with preassigned poles. For such approximants, the following error expression holds:

$$f(z) - \frac{P_n(z)}{Q_n(z)} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{\tilde{\omega}_p(z)\tilde{\pi}_q(z)Q_n(x)}{\tilde{\omega}_p(x)\tilde{\pi}_q(x)Q_n(z)} \frac{f(x)}{x-z}\, \mathrm{d}x \tag{3.3}$$

with $z \in G$ such that $Q_n(z) \neq 0$ and $\Gamma = \partial G$, the boundary of G.

After this general outline, we shall restrict our attention to the function (1.1), i.e.,

$$f(z) = F_\mu(z) = \int_{-\pi}^{\pi} D(t,z)\, \mathrm{d}\mu(\theta), \quad t = \mathrm{e}^{\mathrm{i}\theta}$$

which is analytic in $\hat{\mathbb{C}} - \mathbb{T} = G$. We shall take $\alpha = \{\alpha_k\}_0^\infty \subset \mathbb{D}$ $(\alpha_0 = 0)$ and $\beta = \hat{\alpha} = \{\hat{\alpha}_k\}_0^\infty \subset \mathbb{E}$ with $\hat{\alpha}_k = 1/\overline{\alpha}_k$, $k = 0, 1, \ldots$. Accordingly, we replace $\tilde{\omega}_k$ and $\tilde{\pi}_k$ by

$$\omega_k(z) = \prod_{j=1}^{k} (z - \alpha_j) \quad \text{and} \quad \pi_k(z) = \prod_{j=1}^{k} (1 - \overline{\alpha}_j z), \quad k = 1, 2, \ldots$$

and write the factor $z - \alpha_0 = z$ explicitly. If $Q_n(z)$ is a polynomial with all its zeros in $\mathbb{T}$, we have from (3.3),

$$F_\mu(z) - F_n(z) = F_\mu(z) - \frac{P_n(z)}{Q_n(z)} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{z\omega_p(z)\pi_q(z)Q_n(x)}{x\omega_p(x)\pi_q(x)Q_n(z)} \frac{F_\mu(x)}{x-z}\, \mathrm{d}x \tag{3.4}$$

for $z \in \mathrm{Int}(\Gamma)$, Γ being any Jordan curve contained in $\hat{\mathbb{C}} - \mathbb{T}$. Eq. (3.4) gives the error expression for the MPTA $F_n(z)$ of order $(p+1, q+1)$, $p+q = n-1$, with preassigned denominator $Q_n(z)$ with zeros in $\mathbb{T}$.

By application of Fubini's theorem and having (1.1) in mind, we can rewrite (3.4) for $z \in \mathrm{Int}(\Gamma)$ as

$$F_\mu(z) - F_n(z) = \frac{z\omega_p(z)\pi_q(z)}{Q_n(z)} \frac{1}{2\pi \mathrm{i}} \int_{-\pi}^{\pi} \left[\int_\Gamma \frac{Q_n(x)}{x\omega_p(x)\pi_q(x)} \frac{t+x}{t-x} \frac{\mathrm{d}x}{x-z} \right] \mathrm{d}\mu(\theta), \quad t = \mathrm{e}^{\mathrm{i}\theta}. \tag{3.5}$$

To fix the ideas, we can consider $\Gamma = T_r \cup T_R$ with $0 < r < 1 < R$ where for $\delta > 0$, $T_\delta = \{z \in \mathbb{C} : |z| = \delta\}$ so that in this case the interior of Γ is the exterior of an annulus $\mathrm{Int}(\Gamma) = \{z \in \mathbb{C} : |z| < r\} \cup \{z \in \mathbb{C} : |z| > R\}$. If we denote by Γ^- the curve Γ, oriented in negative sense, then (3.5) becomes with this choice of Γ,

$$F_\mu(z) - F_n(z) = -\frac{z\omega_p(z)\pi_q(z)}{Q_n(z)} \int_{-\pi}^{\pi} \left[\frac{1}{2\pi \mathrm{i}} \int_{\Gamma^-} \frac{Q_n(x)}{x\omega_p(x)\pi_q(x)} \frac{t+x}{t-x} \frac{\mathrm{d}x}{x-z} \right] \mathrm{d}\mu(\theta), \quad t = \mathrm{e}^{\mathrm{i}\theta}. \tag{3.6}$$

If we assume that α is compactly contained in $\mathbb{D}$, then it is always possible to choose r and R such that the function

$$g(x) = \frac{Q_n(x)}{x\omega_p(x)\pi_q(x)} \frac{t+x}{x-z}$$

is analytic in $\mathrm{Int}(\Gamma^-) = \{z \in \mathbb{C} : r < |z| < R\}$. Hence, using the Cauchy integral formula, we obtain

$$F_\mu(z) - F_n(z) = \frac{2z\omega_p(z)\pi_q(z)}{Q_n(z)} \int_{-\pi}^{\pi} \frac{Q_n(t)}{\omega_p(t)\pi_q(t)} \frac{\mathrm{d}\mu(\theta)}{t-z}, \quad t = \mathrm{e}^{\mathrm{i}\theta}. \tag{3.7}$$

This formula was also derived in [11] in a different way.

Remark. For a general MRA in the strong sense of order $(p+1, q+1)$, it is sometimes said that in this situation it has order $p+1$ in $\mathbb{D}$ and order $q+1$ in $\mathbb{E}$. By a balanced order $(p+1, q+1)$ we mean that $|p-q| \leqslant 1$, both in the weak and in the strong sense. Since, in general, $p+q \leqslant 2n-1$, the maximal balanced order is obtained for the order $(n, n+1)$ and $(n+1, n)$, in which case we have balanced MPAs. In the sequel, we shall be concerned with balanced approximants that are one interpolation condition short of being a balanced MPA. These are called (multipoint) modified approximants (MA).

In order to investigate MRAs with balanced order, it will be convenient to recall the close connection between MRAs to F_μ and R-Szegő quadrature formulas. This is similar to the connection that exists between Gauss–Christoffel formulas and Padé approximants to a Stieltjes function when the measure is supported on the real line.

Let $\mathscr{G}$ be the set of all regions G (closed and connected) in $\mathbb{C}$ such that $\mathbb{T} \subset G$, $0 \notin G$ and $G \cap \{\alpha \cup \hat{\alpha}\} = \emptyset$. Suppose that $\Gamma = \partial G$ is a finite union of Jordan curves. Suppose $G \in \mathscr{G}$ and suppose that f is a function analytic in G. From Cauchy's theorem, it then follows that

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{z+x}{z-x} \left(-\frac{f(x)}{2x} \right) \mathrm{d}x \tag{3.8}$$

whenever z is interior in G. From (3.8) and Fubini's theorem, it results that

$$I_\mu\{f\} = \int_{-\pi}^{\pi} f(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\mu(\theta) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma F_\mu(x) \left(-\frac{f(x)}{2x} \right) \mathrm{d}x. \tag{3.9}$$

Let $F_n(z)$ be a rational function of type (n, n) with n distinct poles $\{x_{j,n} : j = 1, \ldots, n\}$ on $\mathbb{T}$, then we have the simple partial fraction decomposition

$$F_n(z) = \frac{P_n(z)}{Q_n(z)} = \lambda_n + \sum_{j=1}^{n} A_{j,n} \frac{x_{j,n}+z}{x_{j,n}-z} = \lambda_n + \sum_{j=1}^{n} A_{j,n} D(x_{j,n}, z). \tag{3.10}$$

Set

$$I_n\{f\} = \sum_{j=1}^{n} A_{j,n} f(x_{j,n}). \tag{3.11}$$

Using the Newton interpolation formula plus error term for $D(t,z)$, namely (see [7]),

$$\frac{t+z}{t-z} = 1 + 2\sum_{k=1}^{m} \frac{z\omega_{k-1}(z)}{\omega_k(t)} + 2\frac{z\omega_m(z)}{(t-z)\omega_m(t)} \tag{3.12}$$

and

$$\frac{t+z}{t-z} = -1 - 2\sum_{k=1}^{m} \frac{t^k \pi_{k-1}(z)}{z^k \pi_k(t)} + 2\frac{t^{m+1}\pi_m(z)}{z^m(t-z)\pi_m(t)}, \quad t \in \mathbb{T} \tag{3.13}$$

((3.13) is deduced from ((3.12) by using a substar conjugate), we obtain the following lemma (see [11] for the case $p+q=n-1$).

Lemma 3.1. *Let F_n be as in* (3.10) *and $I_n\{f\}$ as in* (3.11). *Then the following statements are equivalent*

(i) *$F_n(z)$ given by* (3.10) *is a MRA to $F_\mu(z)$ of order $(p+1, q+1)$ in the strong sense.*

(ii) *$I_n\{f\}$ given by* (3.11) *is exact in $\mathscr{L}_{p,q}$.*

From this lemma, we immediately get the following corollary.

Corollary 3.2. *Let F_n be as in* (3.10) *and $I_n\{f\}$ as in* (3.11). *Suppose $I_n\{f\}$ is exact in $\mathscr{L}_{p,q}$. Then, for any function f, analytic in $G \in \mathscr{G}$ and $\Gamma = \partial G$:*

$$I_n\{f\} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma F_n(x) \left(-\frac{f(x)}{2x}\right) \mathrm{d}x. \tag{3.14}$$

By (3.14) we immediately get an error expression for such a quadrature:

$$I_\mu\{f\} - I_n\{f\} = \frac{1}{2\pi \mathrm{i}} \int_\Gamma [F_\mu(x) - F_n(x)] \left(-\frac{f(x)}{2x}\right) \mathrm{d}x \tag{3.15}$$

when f is analytic in G. By Theorem 2.2 and Lemma 3.1, we have immediately the following corollary.

Corollary 3.3. (1) *There cannot exist a balanced MRA for F_μ of maximal balanced order* (*i.e.*, *a MPA of balanced order*) *with poles on $\mathbb{T}$.*

(2) *The only MRAs for F_μ of balanced order (n,n) in the strong sense are those whose poles are the zeros of the paraorthogonal functions $\phi_n(z) + \tau_n \phi_n^*(z)$, $\tau_n \in \mathbb{T}$ with respect to μ.*

The MRAs of order (n,n) addressed in the last corollary are the ones that we shall concentrate on in the sequel. They depend on a parameter $\tau_n \in \mathbb{T}$ and they are called modified approximants (MA). We shall denote them as $R_n(z,\tau_n)$. The following representation holds (see [12]):

$$R_n(z,\tau_n) = -\frac{\psi_n(z) - \tau_n \psi_n^*(z)}{\phi_n(z) + \tau_n \phi_n^*(z)}, \tag{3.16}$$

where ϕ_n are the orthogonal rational functions for μ and ψ_n are the associated functions of the second kind.

We first give a result concerning uniform convergence of these MA (see also [3]).

Theorem 3.4. *Suppose* $\sum_{k=1}^{\infty}(1-|\alpha_k|)=\infty$. *Then the sequence of MA* $\{R_n(z,\tau_n)\colon n=1,2,\ldots;\tau_n\in\mathbb{T}\}$ *given by* (3.16) *converges locally uniformly to* $F_\mu(z)$ *in* $\hat{\mathbb{C}}-\mathbb{T}$.

Proof. Let $\{x_{j,n}\}_0^n$ be the zeros of $Q_n(z,\tau_n)$ and let $I_n\{f\}=\sum_{j=1}^n A_{j,n}f(x_{j,n})$ be the corresponding n-point R-Szegő quadrature formula. By definitions (3.10) and (3.11) we can write for all $z\in\mathbb{D}\cup\mathbb{E}$:

$$R_n(z,\tau_n)=\sum_{j=1}^{n}A_{j,n}D(x_{j,n},z)=I_n\{D(t,z)\}. \tag{3.17}$$

Since $\{I_n\{f\}\colon n=1,2,\ldots\}$ converges for any integrable function $f\in R_\mu(\mathbb{T})$ by Theorem 2.3, we deduce that $\{R_n(z,\tau_n)\colon n=1,2,\ldots\}$ converges pointwise to $I_\mu\{D(t,z)\}=F_\mu(z)$. For z in a compact subset of $\mathbb{D}$, the rest of the proof is an immediate consequence of the Stieltjes–Vitali theorem [18] because from (3.17) we see that $\{R_n(z,\tau_n)\}$ is a normal family. For z in a compact subset of $\mathbb{E}$, the result is obtained by considering substar conjugates. □

The rate of convergence of these MA will be discussed in Section 5.

For the error $E_n(z,\tau_n)=F_\mu(z)-R_n(z,\tau_n)$ we draw the following expression from [12], which is readily obtained from (2.2) and (2.3):

$$E_n(z,\tau_n)=\frac{1}{f(z)Q_n(z,\tau_n)}\int_{-\pi}^{\pi}D(t,z)f(t)Q_n(t,\tau_n)\,\mathrm{d}\mu(\theta),\quad t=\mathrm{e}^{\mathrm{i}\theta}, \tag{3.18}$$

where $Q_n(z,\tau)=\phi_n(z)+\tau\phi_n^*(z)$ and f is any function in $\mathscr{L}_{(n-1)*}\cap\mathscr{L}_{n*}(1/\overline{\alpha}_n)$ which is not identically zero. By Lemma 3.1 we can also make use of (3.7) to get

$$E_n(z,\tau_n)=\frac{2z\omega_p(z)\pi_q(z)}{\chi_n(z,\tau_n)}\int_{-\pi}^{\pi}\frac{\chi_n(t,\tau_n)}{\omega_p(t)\pi_q(t)}\frac{\mathrm{d}\mu(\theta)}{t-z},\quad t=\mathrm{e}^{\mathrm{i}\theta}, \tag{3.19}$$

p and q being nonnegative integers with $p+q=n-1$ and $\chi_n(z,\tau_n)$ defined by $Q_n(z,\tau_n)=\chi_n(z,\tau_n)/\pi_n(z)$. This formula does not explicitly show the interpolation properties of the approximants. These will come out more clearly in the formulas derived in the following theorem.

Theorem 3.5. *If* $R_n(z,\tau_n)$ *is the MA to* F_μ *of* (3.16), *then for all* $z\in\mathbb{D}\cup\mathbb{E}$ *and* $n>2$, *the approximation error is given by*

$$E_n(z,\tau_n)=\frac{2z\omega_{n-1}(z)\pi_{n-1}(z)}{\chi_n(z,\tau_n)}\int_{-\pi}^{\pi}\frac{\chi_n(t,\tau_n)}{\pi_{n-1}(t)\omega_{n-1}(t)}\frac{\mathrm{d}\mu(\theta)}{t-z} \tag{3.20}$$

$$=\frac{2\omega_{n-1}(z)\pi_{n-1}(z)}{z^{n-2}\chi_n(z,\tau_n)}\int_{-\pi}^{\pi}\frac{\chi_n(t,\tau_n)t^{n-1}}{\omega_{n-1}(t)\pi_{n-1}(t)}\frac{\mathrm{d}\mu(\theta)}{t-z}, \tag{3.21}$$

where $t=\mathrm{e}^{\mathrm{i}\theta}$ *and* $\chi_n(z,\tau_n)=Q_n(z,\tau_n)\pi_n(z)$.

Proof. For the first formula, we choose $f(z) = (1 - \overline{\alpha}_n z)/(z - \alpha_{n-1}) \in \mathscr{L}_{(n-1)*} \cap \mathscr{L}_{n*}(1/\overline{\alpha}_n)$ in (3.18) to get

$$E_n(z, \tau_n) = \frac{z - \alpha_{n-1}}{(1 - \overline{\alpha}_n z) Q_n(z, \tau_n)} \int_{-\pi}^{\pi} Q_n(t, \tau_n) D(t, z) \frac{1 - \overline{\alpha}_n t}{t - \alpha_{n-1}} \, \mathrm{d}\mu(\theta).$$

Next we replace $D(t,z)$ by (3.12) with $m = n - 2$ and use orthogonality properties of $Q_n(z, \tau_n)$ to find

$$E_n(z, \tau_n) = \frac{(z - \alpha_{n-1}) 2z\omega_{n-2}(z)}{(1 - \overline{\alpha}_n z) Q_n(z, \tau_n)} \int_{-\pi}^{\pi} \frac{1 - \overline{\alpha}_n t}{(t - \alpha_{n-1})\omega_{n-2}(t)} \frac{Q_n(t, \tau_n)}{t - z} \, \mathrm{d}\mu(\theta), \quad t = \mathrm{e}^{\mathrm{i}\theta}.$$

Since $Q_n(z, \tau_n) = \chi_n(z, \tau_n)/\pi_n(z)$, the proof of the first formula follows.

For the second formula, we use (3.19) with $p = n - 1$ (hence $q = 0$) and (3.13) with $m = n - 1$. Taking into account that

$$\frac{2}{t - z} = \frac{1}{t}[D(t, z) + 1]$$

and using the orthogonality properties of $Q_n(z, \tau_n)$, the second formula is obtained. $\square$

Remarks. 1. From (3.22), it is not immediately clear that $\lim_{z \to \infty} E_n(z, \tau_n) = 0$ since the integral in (3.20) tends to zero as $z \to \infty$, but the preceding factor tends to infinity. The interpolation at infinity is displayed by the second formula (3.21). Thus, we need the two formulas to display all the interpolation properties. Therefore, we give in Theorem 3.6 below yet another formula for the error.

2. Formula (3.21) could also be obtained from (3.18) by using

$$f(t) = \frac{(1 - \overline{\alpha}_n t)\pi_{n-2}(t)}{\omega_{n-1}(t)}.$$

3. When all $\alpha_k = 0$, then the MPA become two-point Padé approximants (2PA). The latter were studied by Jones et al. [19] in connection with the trigonometric moment problem and continued fractions. Estimation of the error and rates of convergence of these 2PA were given in a recent paper [8].

The formula below shows all the interpolation properties. It shall be used in Section 5.

Theorem 3.6. *For the error of the MA $R_n(z, \tau_n)$, we have for all $z \in \mathbb{D} \cup \mathbb{E}$ and $n > 2$*

$$E_n(z, \tau_n) = \frac{2z\omega_{n-1}(z)\pi_{n-1}(z)}{\chi_n^2(z)} \left[\int_{-\pi}^{\pi} \frac{\chi_n^2(t) \, \mathrm{d}\mu(\theta)}{\pi_{n-1}(t)\omega_{n-1}(t)(t - z)} + \delta_n \right], \tag{3.22}$$

where

$$\delta_n = \int_{-\pi}^{\pi} Q_n(t, \tau_n)(1 - \overline{\alpha}_n t) \, \mathrm{d}\mu(\theta), \quad t = \mathrm{e}^{\mathrm{i}\theta}, \tag{3.23}$$

and $Q_n(z, \tau_n) = \chi_n(z)/\pi_n(z)$.

Proof. We recall the error expression given in (3.20). Without loss of generality (see (3.22)), we may assume that $\chi_n(z)$ is monic. Thus,

$$\frac{\chi_n(t)-\chi_n(z)}{t-z}=t^{n-1}+P(t),\quad P\in\Pi_{n-2},$$

so that

$$\frac{\chi_n(t)-\chi_n(z)}{\omega_{n-1}(t)\pi_{n-1}(t)(t-z)}=\frac{t^{n-1}+P(t)}{\pi_{n-1}(t)\omega_{n-1}(t)}=\frac{(1-\overline{\alpha}_n t)[t^{n-1}+P(t)]}{\pi_n(t)\omega_{n-1}(t)}.$$

Therefore,

$$\chi_n(t)\left[\frac{\chi_n(t)-\chi_n(z)}{\pi_{n-1}(t)\omega_{n-1}(t)(t-z)}\right]=Q_n(t,\tau_n)\frac{(1-\overline{\alpha}_n t)[t^{n-1}+P(t)]}{\omega_{n-1}(t)}$$

and also

$$\int_{-\pi}^{\pi}\chi_n(t)\left[\frac{\chi_n(t)-\chi_n(z)}{\pi_{n-1}(t)\omega_{n-1}(t)(t-z)}\right]\mathrm{d}\mu(\theta)=\int_{-\pi}^{\pi}\frac{Q_n(t,\tau_n)t^{n-1}(1-\overline{\alpha}_n t)}{\omega_{n-1}(t)}\mathrm{d}\mu(\theta)$$
$$+\int_{-\pi}^{\pi}Q_n(t,\tau_n)\frac{P(t)(1-\overline{\alpha}_n t)}{\omega_{n-1}(t)}\mathrm{d}\mu(\theta).$$

The second term on the right-hand side equals zero by the paraorthogonality of Q_n. Setting the first term equal to δ_n, we have by (3.20) the desired form for the error.

It remains to prove (3.23). Since $t^{n-1}=\omega_{n-1}(t)+\sum_{j=0}^{n-2}\gamma_j\omega_j(t)$, we find by the paraorthogonality of Q_n that (3.23) holds indeed. □

Remark. The presence of the "strange" term δ_n in (3.22) is due to the deficiencies in the orthogonality properties of Q_n. For a Stieltjes function with a measure on the real line, true orthogonal instead of paraorthogonal functions are used and this term will not appear, cf. with e.g. [23].

To end this section, we give a result concerning the existence of MPA with balanced order, i.e., order $(n,n+1)$ or $(n+1,n)$. It can be found in [5]. For a proof see [3]. It expresses the interpolation of $R_n=-\psi_n/\phi_n$ in the points $0=\alpha_0,\alpha_1,\ldots,\alpha_{n-1},\hat{\alpha}_0,\ldots,\hat{\alpha}_n$ and of $R_n^{\times}=\psi_{n*}/\phi_{n*}$ in the points $\alpha_0,\ldots,\alpha_n,\hat{\alpha}_0,\ldots,\hat{\alpha}_{n-1}$.

Theorem 3.7. *With the notations introduced before, we have*
(1) $B_n(z)[F_\mu(z)+\psi_n(z)/\phi_n(z)]=\hat{h}(z)$, *$\hat{h}$ analytic in $\mathbb{E}$ and* $\lim_{z\to\infty}\hat{h}(z)=0$.
(2) $B_{n-1}(z)[F_\mu(z)\phi_{n*}(z)-\psi_{n*}(z)]=\hat{g}(z)$, *$\hat{g}$ analytic in $\mathbb{E}$ and* $\lim_{z\to\infty}\hat{g}(z)=0$.
(3) $[B_{n-1}(z)]^{-1}[F_\mu(z)\phi_n(z)+\psi_n(z)]=h(z)$, *$h$ analytic in $\mathbb{D}$ and* $h(0)=0$.
(4) $[B_{n-1}(z)]^{-1}[F_\mu(z)-\psi_{n*}(z)/\phi_{n*}(z)]=g(z)$, *$g$ analytic in $\mathbb{D}$ and* $g(0)=0$.

This implies

Corollary 3.8. *Defining $R_n(z)=-\psi_n(z)/\phi_n(z)$ and $R_n^{\times}(z)=\psi_{n*}(z)/\phi_{n*}(z)=\psi_n^*(z)/\phi_n^*(z)$, we have:*
1. *$R_n(z)$ is an analytic function in $\mathbb{E}$ and it is the MPA to $F_\mu(z)$ of order $(n,n+1)$ in the weak sense.*

2. *$R_n^\times(z)$ is an analytic function in $\mathbb{D}$ and it is the MPA to $F_\mu(z)$ of order $(n+1,n)$ in the weak sense.*

In [5], it was proved that the sequence $\{R_n^\times(z)\}$ converges to $F_\mu(z)$ locally uniformly in $\mathbb{D}$ and the sequence $\{R_n\}$ converges to the same function locally uniformly in $\mathbb{E}$. In Section 5 we give estimates of the rate of convergence for both these sequences.

We close this section by deriving also for the approximant R_n an expression for the approximation error which clearly reflects the interpolation properties of R_n. An expression for the error of $R_n^\times$ can be obtained similarly or derived from the one below by substar conjugation.

Theorem 3.9. *Let $R_n = -\psi_n/\phi_n$ denote the MPA as defined above, with approximation error*

$$E_n(z) = F_\mu(z) - R_n(z) = F_\mu(z) + \frac{\psi_n(z)}{\phi_n(z)}.$$

Then for $z \in \mathbb{D} \cup \mathbb{E}$

$$E_n(z) = \frac{2z\omega_{n-1}(z)}{\pi_n(z)\phi_n^2(z)} \int_{-\pi}^{\pi} \frac{\phi_n^2(t)\pi_n(t)}{(t-z)\omega_{n-1}(t)}\,\mathrm{d}\mu(\theta), \quad t = \mathrm{e}^{\mathrm{i}\theta}. \tag{3.24}$$

Proof. By setting $\tau_n = 0$ in (3.18) we find

$$E_n(z) = \frac{1}{\phi_n(z)f(z)} \int_{-\pi}^{\pi} D(t,z)\phi_n(t)f(t)\,\mathrm{d}\mu(\theta), \quad t = \mathrm{e}^{\mathrm{i}\theta}.$$

Since

$$D(t,z) = \frac{t+z}{t-z} = 1 + \frac{2z}{t-z},$$

we can write

$$E_n(z) = \frac{2z}{\phi_n(z)f(z)} \int_{-\pi}^{\pi} \frac{\phi_n(t)f(t)}{t-z}\,\mathrm{d}\mu(\theta), \quad t = \mathrm{e}^{\mathrm{i}\theta}. \tag{3.25}$$

Let $\phi_n(z) = P_n(z)/\pi_n(z)$, $P_n \in \Pi_n$, then

$$\frac{P_n(t) - P_n(z)}{t-z} = P(z) \in \Pi_{n-1}.$$

Thus,

$$\frac{P_n^2(t)f(t)}{\pi_n(t)(t-z)} - \frac{\phi_n(t)P_n(z)f(t)}{t-z} = \phi_n(t)P(z)f(t).$$

We now choose $f(t) = 1/\omega_{n-1}(t) \in \mathscr{L}_{(n-1)*}$, then by the orthogonality properties of ϕ_n, we get

$$\int_{-\pi}^{\pi} \frac{\phi_n(t)f(t)}{t-z}\,\mathrm{d}\mu(\theta) = \frac{1}{P_n(z)} \int_{-\pi}^{\pi} \frac{\phi_n^2(t)\pi_n(t)}{(t-z)\omega_{n-1}(t)}\,\mathrm{d}\mu(\theta), \quad t = \mathrm{e}^{\mathrm{i}\theta},$$

which finally yields the desired expression. $\square$

Remarks. 1. The error (3.24) for these maximal balanced order approximants (MPA) R_n based on the orthogonal rational functions should be compared with the error (3.22) for the nearly maximal balanced order approximants (MA) based on the paraorthogonal rational functions.

2. Note that (3.24) displays all the interpolation conditions, i.e., if we could assure that for a given n, ϕ_n does not vanish in the points $\alpha_0 = 0, \alpha_1, \ldots, \alpha_{n-1}$, then we see that $R_n(\alpha_k) = F_\mu(\alpha_k)$, $k = 0, 1, \ldots, n-1$ and

$$R_n(1/\bar{\alpha}_k) = F_\mu(1/\bar{\alpha}_k), \quad k = 1, \ldots, n; \qquad \lim_{z \to \infty} [R_n(z) - F_\mu(z)] = 0.$$

4. Asymptotics for orthogonal rational functions

In this section we consider the convergence of the orthogonal rational functions $\phi_n(z)$ and of the quasiorthogonal rational functions $Q_n(z, \tau_n)$. We give results about ratio asymptotics for the ϕ_n, which implies nth root asymptotics for the ϕ_n and the Q_n. This will lead in the next section to estimates of the rate of convergence of sequences of modified approximants to the function $F_\mu(z)$ and of quadrature formulas.

To formulate the theorems, we need to consider the following conditions for the measure:

(S) $\int_{-\pi}^{\pi} \log \mu'(\theta)\,\mathrm{d}\theta > -\infty$ (Szegő condition),

(R) $\mu' > 0$ a.e. (Rakhmanov condition)

and the following conditions for the basic points α_k

(B) $\sum_{n=1}^{\infty}(1 - |\alpha_k|) = \infty$ (Blaschke condition)

(C) The set α is compactly contained in $\mathbb{D}$ (compactly contained condition).

Remarks. 1. The condition $\log \mu' \in L_1$ is the well-known Szegő condition. If it is satisfied, the spectral factor

$$\sigma(z) = \exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi} D(t,z) \log \mu'(\theta)\,\mathrm{d}\theta\right\}, \quad t = \mathrm{e}^{\mathrm{i}\theta}$$

is well defined.

2. The weaker condition (R) was considered by Erdős for the interval $[-1, 1]$. Erdős and Turán used it to prove convergence in L_2. Therefore, it is sometimes called the Erdős condition. Rakhmanov used this condition to prove that the recurrence coefficients of the associated orthogonal polynomials behave asymptotically as the recurrence coefficients of the Chebyshev polynomials.

3. The condition (B) implies that the Blaschke product diverges, i.e., $B_n(z)$ goes to zero locally uniformly in $\mathbb{D}$. Obviously, condition (C) implies condition (B).

We first recall some results about ratio asymptotics which can be found in Li and Pan [20] when conditions (S) and (B) hold. For the case where conditions (S) and (C) hold, Theorem 4.1 also easily follows from Theorem 6.12 in [6].

Theorem 4.1. *Assume that conditions* (S) *and* (B) *hold and let ϕ_n be the orthonormal rational functions for $\mathcal{L}_n$ and $\sigma(z)$ the spectral factor of μ. Then*

$$\lim_{n\to\infty} \frac{\phi_n^*(z)(1 - \bar{\alpha}_n z)}{\phi_n^*(0)} = \frac{\sigma(0)}{\sigma(z)} \quad \textit{locally uniformly in } \mathbb{D}.$$

Theorem 4.2. *Assume that conditions* (S) *and* (B) *hold and let* $k_n(z,w) = \sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(w)}$ *be the reproducing kernel for* $\mathscr{L}_n$ *and denote the Szegő kernel for* $H_2(\mathbb{D})$ *as*

$$s(z,w) = [(1 - z\overline{w}z)\sigma(z)\overline{\sigma(w)}]^{-1}.$$

Then we have

$$\lim_{n\to\infty} k_n(z,w) = s(z,w) \quad \textit{locally uniformly in } \mathbb{D} \times \mathbb{D}.$$

These theorems imply the following result.

Theorem 4.3. *Assume conditions* (S) *and* (C) *hold. Then for the orthonormal rational functions* ϕ_n *we have the following local uniform convergence in the indicated regions*:

$$\lim_{n\to\infty} \frac{\phi_n(z)}{\phi_n^*(z)} = 0, \quad z \in \mathbb{D} \quad \textit{and} \quad \lim_{n\to\infty} \frac{\phi_n^*(z)}{\phi_n(z)} = 0, \quad z \in \mathbb{E}.$$

Proof. The second part follows from the first part by taking superstar conjugates. Thus, we only have to prove the first part. By Theorem 4.2 we have

$$\lim_{n\to\infty} k_n(z,z) = s(z,z) \quad \text{locally uniformly in } \mathbb{D}.$$

But, since $k_n(z,z) = \sum_{k=0}^{n} |\phi_k(z)|^2$, and since $\sum_{k=0}^{\infty} |\phi_k(z)|^2$ converges locally uniformly in $\mathbb{D}$, we get $\lim_{n\to\infty} \phi_n(z) = 0$ locally uniformly in $\mathbb{D}$. In combination with Theorem 4.1 this gives

$$\lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)(1 - \overline{\alpha}_n z)} = 0.$$

Since α is compactly contained in $\mathbb{D}$, there exists some R, $0 < R < 1$, such that

$$\frac{1}{2} \leqslant \frac{1}{|1 - \overline{\alpha}_n z|} \leqslant \frac{1}{1 - R}.$$

Thus,

$$\lim_{n\to\infty} \frac{\phi_n(z)\phi_n^*(0)}{\phi_n^*(z)} = 0. \tag{4.1}$$

Now, using the Christoffel–Darboux formula [3]

$$\sum_{k=0}^{n} \phi_k(z)\overline{\phi_k(w)} = \frac{\phi_n^*(z)\overline{\phi_n^*(w)} - \zeta_n(z)\overline{\zeta_n(w)}\phi_n(z)\overline{\phi_n(w)}}{1 - \zeta_n(z)\overline{\zeta_n(w)}}$$

for $z = w = 0$, we find

$$|\phi_n^*(0)|^2 - |\zeta_n(0)|^2|\phi_n(0)|^2 = (1 - |\zeta_n(0)|^2)\sum_{k=0}^{n} |\phi_k(0)|^2. \tag{4.2}$$

Thus, because there exists an $R < 1$ such that $|\alpha_n| \leqslant R$, for all $n = 1,2,\ldots$

$$|\phi_n^*(0)|^2 \geqslant (1 - |\zeta_n(0)|^2)|\phi_0(0)|^2 = 1 - |\alpha_n|^2 \geqslant 1 - R^2 = A > 0, \quad n = 1,2,\ldots. \tag{4.3}$$

The proof then follows from (4.1) and (4.3). □

Remark. In [21, Theorem 2.3], Pan proved that under the weaker condition (R) for the measure and (B) for the basic points one has

$$\lim_{n\to\infty} \frac{\phi_n^*(z)(1-\overline{\alpha}_n z)}{\phi_n(z)(z-\alpha_n)} = 0 \quad \text{locally uniformly in } \mathbb{E}.$$

Without assuming condition (S) we were not able to remove the factor $(1-\overline{\alpha}_n z)/(z-\alpha_n)$ from the left-hand side.

For the ratio asymptotics of the sequence $\{\phi_n(z)\}$ we have another theorem by Pan [21].

Theorem 4.4. *Let conditions* (R) *and* (B) *hold. Then for the orthonormal functions ϕ_n we have locally uniform convergence in the indicated regions*

1. $\lim_{n\to\infty} \dfrac{(1-\overline{\alpha}_{n+1}z)\phi_{n+1}^*(z)\phi_n^*(0)}{(1-\overline{\alpha}_n z)\phi_n^*(z)\phi_{n+1}^*(0)} = 1,\ z\in\mathbb{D}$
2. $\lim_{n\to\infty} -\dfrac{\alpha_{n+1}}{|\alpha_{n+1}|}\dfrac{(1-\overline{\alpha}_{n+1}z)\phi_{n+1}(z)\overline{\phi_n^*(0)}}{(z-\alpha_n)\phi_n(z)\overline{\phi_{n+1}^*(0)}} = 1,\ z\in\mathbb{E}.$

Proof. The first formula is Theorem 2.2 in Pan [21] and the second formula is again obtained by taking substar conjugates in the first one. □

Remark. 1. If condition (R) is replaced by condition (S) in the previous theorem, then the result follows immediately from Theorem 4.1.

2. When all $\alpha_k = 0$, then the first part of Theorem 4.4 becomes

$$\lim_{n\to\infty} \frac{\phi_{n+1}^*(z)\phi_n^*(0)}{\phi_n^*(z)\phi_{n+1}^*(0)} = 1.$$

But in the polynomial case, we also have $\lim_{n\to\infty} \phi_{n+1}^*(0)/\phi_n^*(0) = 1$ (see, e.g., [21, Lemma 3.4]) and thus we have then $\lim_{n\to\infty} \phi_{n+1}^*(z)/\phi_n^*(z) = 1$ locally uniformly in $\mathbb{D}$. Similarly, Theorem 4.4(2) implies that in the polynomial case $\lim_{n\to\infty} \phi_{n+1}(z)/\phi_n(z) = z$ locally uniformly in $\mathbb{E}$.

From Theorem 4.4 we can deduce the nth root asymptotics for ϕ_n^*.

Theorem 4.5. *Suppose conditions* (R) *and* (C) *hold. Then*

$$\lim_{n\to\infty} |\phi_n^*(z)|^{1/n} = 1 \quad \textit{locally uniformly in } \mathbb{D}.$$

Proof. Since in general $a_{n+1}/a_n \to 1$ implies $a_n^{1/n} \to 1$, we have by Theorem 4.4(1),

$$\lim_{n\to\infty} \left|\frac{\phi_n^*(z)(1-\overline{\alpha}_n z)}{\phi_n^*(0)}\right|^{1/n} = 1. \tag{4.4}$$

From (4.3), we know that there exists a positive number A such that

$$|\phi_n^*(0)| \geqslant A > 0, \quad n = 1,2,\ldots. \tag{4.5}$$

From (4.2) and since $|\zeta_n(0)|^2 = |\alpha_n|^2$, we have

$$|\phi_n^*(0)| \leqslant |\alpha_n|^2 |\phi_n(0)|^2 + (1-|\alpha_n|^2) \sum_{k=0}^{\infty} |\phi_k(0)|^2.$$

Since $\sum_{k=0}^{\infty} |\phi_k(0)|^2 < \infty$, there exists a positive B such that $|\phi_n^*(0)| \leqslant B$. In combination with (4.5) this implies that

$$\lim_{n\to\infty} |\phi_n^*(0)|^{1/n} = 1. \tag{4.6}$$

Furthermore, for any $z \in \mathbb{D}$, there exists an $M > 0$ such that $M \leqslant |1-\overline{\alpha}_n z| \leqslant 2$, which implies that

$$\lim_{n\to\infty} |1-\overline{\alpha}_n z|^{1/n} = 1 \quad \text{locally uniformly in } \mathbb{D}. \tag{4.7}$$

Finally, from (4.4), (4.6) and (4.7), the proof is achieved. □

To obtain root asymptotics for the $|\phi_n|$ we need to know more about the distribution of the points α_k. Let us consider the normalized counting measure ν_n^α defined by

$$\nu_n^\alpha = \frac{1}{n} \sum_{j=1}^{n} \delta_{\alpha_j}$$

which assigns a point mass at α_j, taking into account the multiplicity of α_j. For any measure ν, the logarithmic potential is defined as

$$V_\nu(z) = -\int \log|z-x| \, \mathrm{d}\nu(x).$$

Obviously, we have for $\omega_n(z) = \prod_{j=1}^{n}(z-\alpha_j)$ that

$$|\omega_n(z)|^{1/n} = \exp\{-V_{\nu_n^\alpha}(z)\}.$$

Now assume that that ν_n^α converges to some measure ν^α in the weak star topology, which we denote as $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$. This convergence implies [23]

$$\lim_{n\to\infty} |\omega_n(z)|^{1/n} = \exp\{-V_{\nu^\alpha}(z)\}, \quad z \in \mathbb{C} - \operatorname{supp}(\nu^\alpha)$$

and

$$\limsup_{n\to\infty} |\omega_n(z)|^{1/n} \leqslant \exp\{-V_{\nu^\alpha}(z)\}, \quad z \in \mathbb{C}$$

where convergence is uniform on any compact subset of the indicated region. Set

$$\overline{\omega}_n(z) = \prod_{j=1}^{n} (z-\overline{\alpha}_j) = z^n \pi_n(1/z), \quad z \neq 0.$$

Let $\nu^{\overline{\alpha}}$ be the measure associated with the point set $\overline{\alpha} = \{\overline{\alpha}_n\}_0^\infty$, just like ν^α was associated with the set α. Note that from the definition of $V_{\nu_n^\alpha}$ and $V_{\nu_n^{\overline{\alpha}}}$, we have

$$V_{\nu_n^\alpha}(z) = -\int \log|z-x| \, \mathrm{d}\nu_n^\alpha(x) = -\frac{1}{n} \sum_{j=1}^{n} \log|z-\alpha_j|$$

and

$$V_{\nu_n^{\bar{\alpha}}}(z) = -\int \log|z-x|\,\mathrm{d}\nu_n^{\bar{\alpha}}(x) = -\frac{1}{n}\sum_{j=1}^{n}\log|z-\bar{\alpha}_j| = -\frac{1}{n}\sum_{j=1}^{n}\log|\bar{z}-\alpha_j| = V_{\nu_n^{\alpha}}(\bar{z}).$$

Therefore, $V_{\nu^{\bar{\alpha}}}(z) = V_{\nu^{\alpha}}(\bar{z})$.

The introduction of $\nu^{\bar{\alpha}}$ allows us to state that for any $z^{-1} \in \mathbb{C} - (\{0\} \cup \operatorname{supp}(\nu^{\bar{\alpha}}))$ or equivalently for any $z \in \mathbb{C} - (\{0\} \cup \operatorname{supp}(\nu^{\hat{\alpha}}))$

$$\lim_{n\to\infty} |\pi_n(z)|^{1/n} = |z| \lim_{n\to\infty} |\varpi_n(1/z)|^{1/n} = |z|\exp\{-V_{\nu^{\bar{\alpha}}}(1/z)\} = |z|\exp\{-V_{\nu^{\alpha}}(\hat{z})\}. \tag{4.8}$$

Furthermore, $\lim_{n\to\infty} |\pi_n(0)|^{1/n} = 1$ and

$$\limsup_{n\to\infty} |\pi_n(z)|^{1/n} \leqslant |z|\exp\{-V_{\nu^{\alpha}}(\hat{z})\}, \quad z \in \mathbb{C} - \{0\}. \tag{4.9}$$

All this adds up the the following result.

Lemma 4.6. *Let B_n denote the Blaschke products with zeros $\{\alpha_1, \ldots, \alpha_n\}$. Suppose that for the zero distribution we have $\nu_n^{\alpha} \xrightarrow[n]{*} \nu^{\alpha}$. Then*

$$\lim_{n\to\infty} |B_n(z)|^{1/n} = \exp\{\lambda(z)\} \quad \textit{and} \quad \lim_{n\to\infty} |B_n(z)|^{-1/n} = \exp\{\lambda(\hat{z})\}$$

locally uniformly for $z \in \mathbb{C} - (\{0\} \cup \operatorname{supp}(\nu^{\alpha}) \cup \operatorname{supp}(\nu^{\hat{\alpha}}))$ with

$$\lambda(z) = \int \log|\zeta_z(x)|d\nu^{\alpha}(x), \quad \zeta_z(x) = \frac{x-z}{1-\bar{z}x}. \tag{4.10}$$

For $z \in \mathbb{C} - \{0\}$ we have

$$\limsup_{n\to\infty} |B_n(z)|^{1/n} \leqslant \exp\{\lambda(z)\} \quad \textit{and} \quad \limsup_{n\to\infty} |B_n(z)|^{-1/n} \leqslant \exp\{\lambda(\hat{z})\}$$

Proof. We start with the first limit. For $z \in \mathbb{C} - (\{0\} \cup \operatorname{supp}(\nu^{\alpha}) \cup \operatorname{supp}(\nu^{\hat{\alpha}}))$

$$\lim_{n\to\infty} |B_n(z)|^{1/n} = \lim_{n\to\infty} \frac{|\omega_n(z)|^{1/n}}{|\pi_n(z)|^{1/n}} = |z|^{-1}\exp\{-V_{\nu^{\alpha}}(z) + V_{\nu^{\alpha}}(\hat{z})\}$$

while

$$V_{\nu^{\alpha}}(z) = -\int \log|x-z|d\nu^{\alpha}(x) \quad \text{and} \quad V_{\nu^{\alpha}}(\hat{z}) = -\int \log|1-\bar{z}x|d\nu^{\alpha}(x) + \log|z|,$$

we get the first result.

The second limit is easily derived from the first one since

$$|B_n(1/\bar{z})| = |B_{n*}(z)| = 1/|B_n(z)|.$$

The remaining inequalities follow similarly. $\square$

Note that $\lambda(\hat{z}) = -\lambda(z)$.

For the sequence $\{\phi_n(z)\}$ we have the following result.

Theorem 4.7. *Assume conditions* (R) *and* (C) *hold and let* $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$. *Then we have locally uniformly in the indicated regions*

$$\lim_{n\to\infty} |\phi_n(z)|^{1/n} = \exp\{\lambda(z)\}, \quad z \in \mathbb{E} - \operatorname{supp}(\nu^{\hat{\alpha}}), \tag{4.11}$$

$$\limsup_{n\to\infty} |\phi_n(z)|^{1/n} \leqslant \exp\{\lambda(z)\}, \quad z \in \mathbb{E},$$

where $\lambda(z)$ *is as in* (4.10).

Proof. By taking the substar conjugate in Theorem 4.5 we deduce that

$$\lim_{n\to\infty} \left|\frac{\phi_n(z)}{B_n(z)}\right|^{1/n} = 1 \quad \text{locally uniformly in } \mathbb{E}. \tag{4.12}$$

Thus, the result follows by the previous lemma. □

As a simple illustrative example, we consider the situation where $\lim_{n\to\infty} \alpha_n = a \in \mathbb{D}$ so that $\nu^\alpha = \delta_a$. In this case $\operatorname{supp}(\nu^\alpha) = \{a\}$ and $\operatorname{supp}(\nu^{\bar{\alpha}}) = \{\bar{a}\}$. Furthermore,

$$V_{\delta_a}(z) = -\int \log|z - x| d\delta_a(x) = -\log|z - a|$$

and

$$V_{\delta_{\bar{a}}}(1/z) = -\int \log|1/z - a| d\delta_{\bar{a}} = -\log|1/z - \bar{a}| = \log|z| - \log|1 - z\bar{a}|.$$

Thus, we have uniformly on compact subsets of $\mathbb{E} - \{1/\bar{a}\}$,

$$\begin{aligned}\lim_{n\to\infty} |\phi_n(z)|^{1/n} &= |z|^{-1} \exp\{V_{\nu^{\bar{\alpha}}}(1/z) - V_{\nu^\alpha}(z)\} \\ &= |z|^{-1} \exp\{\log|z| - \log|1 - \bar{a}z| + \log|z - a|\} \\ &= \left|\frac{z - a}{1 - \bar{a}z}\right|.\end{aligned}$$

So when $a = 0$, we have $\lim_{n\to\infty} |\phi_n(z)|^{1/n} = |z|$. In particular, if $\alpha_n = 0$ for all n, then $\phi_n(z)$ becomes the Szegő polynomial and the well-known result that

$$\lim_{n\to\infty} |\phi_n(z)|^{1/n} = |z| \quad \text{locally uniformly in } \mathbb{E}$$

is recovered (cf. [11]).

Next we prove nth root asymptotics for the paraorthogonal rational functions

$$Q_n(z, \tau_n) = \phi_n(z) + \tau_n \phi_n^*(z), \quad \tau_n \in \mathbb{T}.$$

This will be used in Section 5.

Theorem 4.8. *Assume conditions* (S) *and* (C) *hold and let* $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$. *Then we have locally uniformly in the indicated regions*

(1) $\lim_{n\to\infty} |Q_n(z,\tau_n)|^{1/n} = 1,\ z\in\mathbb{D}$.
(2) $\lim_{n\to\infty} |Q_n(z,\tau_n)|^{1/n} = \exp\{\lambda(z)\},\ z\in\mathbb{E}-\operatorname{supp}(v^{\hat{\alpha}})$
(3) $\limsup_{n\to\infty} |Q_n(z,\tau_n)|^{1/n} \leqslant \exp\{\lambda(z)\},\ z\in\mathbb{E}$

Proof. For the first limit, we set

$$\chi_n(z) = \frac{(1-\overline{\alpha}_n z)Q_n(z,\tau_n)}{\phi_n^*(0)}$$

Then for any $z\in\mathbb{D}$

$$\frac{\chi_{n+1}(z)}{\chi_n(z)} = \left[\frac{(1-\overline{\alpha}_{n+1}z)\phi_{n+1}^*(z)\phi_n^*(0)}{(1-\overline{\alpha}_n z)\phi_n^*(z)\phi_{n+1}^*(0)}\right]\left[\frac{\tau_{n+1}+\phi_{n+1}(z)/\phi_{n+1}^*(z)}{\tau_n+\phi_n(z)/\phi_n^*(z)}\right] \equiv \gamma_n\Delta_n.$$

Clearly, $\lim_{n\to\infty}\gamma_n = 1$ by Theorem 4.4, while

$$\frac{1-|\phi_{n+1}(z)/\phi_{n+1}^*(z)|}{1+|\phi_n(z)/\phi_n^*(z)|} \leqslant |\Delta_n| \leqslant \frac{1+|\phi_{n+1}(z)/\phi_{n+1}^*(z)|}{1-|\phi_n(z)/\phi_n^*(z)|}.$$

Since by Theorem 4.3, $\lim_{n\to\infty}\phi_n(z)/\phi_n^*(z) = 0$ locally uniformly in $\mathbb{D}$, we find that $\lim_{n\to\infty}|\Delta_n| = 1$. Thus, we have proved that $\lim_{n\to\infty}|\chi_{n+1}(z)/\chi_n(z)| = 1$ which implies that $\lim_{n\to\infty}|\chi_n(z)|^{1/n} = 1$. Now, since $Q_n(z,\tau_n) = \phi_n^*(0)\chi_n(z)/(1-\overline{\alpha}_n z)$ and $\lim_{n\to\infty}|1-\overline{\alpha}_n z|^{1/n} = 1$ locally uniformly in $\mathbb{D}$ and $\lim_{n\to\infty}|\phi_n^*(0)|^{1/n} = 1$, the proof follows.

For the other two formulas we note that since $\phi_n(z)\neq 0$ in $\mathbb{E}$, we can write

$$Q_n(z,\tau_n) = \phi_n(z)[1+\tau_n\phi_n^*(z)/\phi_n(z)].$$

This gives

$$|\phi_n(z)|[1-|\phi_n^*(z)/\phi_n(z)|] \leqslant |Q_n(z,\tau_n)| \leqslant |\phi_n(z)|[1+|\phi_n^*(z)/\phi_n(z)|]$$

and, consequently,

$$|\phi_n(z)|^{1/n}[1-|\phi_n^*(z)/\phi_n(z)|]^{1/n} \leqslant |Q_n(z,\tau_n)|^{1/n} \leqslant |\phi_n(z)|^{1/n}[1+|\phi_n^*(z)/\phi_n(z)|]^{1/n}.$$

By Theorem 4.5, $\lim_{n\to\infty}\phi_n^*(z)/\phi_n(z) = 0$ locally uniformly in $\mathbb{E}$, which yields immediately $\lim_{n\to\infty}|\phi_n(z)|^{1/n} = \lim_{n\to\infty}|Q_n(z,\tau_n)|^{1/n}$. By Theorem 4.7, the proof is completed. □

Let us now see how the sequence $\{Q_n(z,\tau_n)\}$ behaves on the boundary $\mathbb{T}$. We define

$$\|Q_n\|_\infty = \max_{z\in\mathbb{T}}|Q_n(z,\tau_n)| = \max_{z\in\mathbb{D}\cup\mathbb{T}}|Q_n(z,\tau_n)|.$$

Theorem 4.9. *Assume conditions* (S) *and* (C) *hold and let* $v_n^\alpha \xrightarrow[n]{*} v^\alpha$. *Then*

$$\lim_{n\to\infty}\|Q_n\|_\infty^{1/n} = 1.$$

Proof. Take $z\in\mathbb{D}$, then $|Q_n(z,\tau_n)| \leqslant \|Q_n\|_\infty$ and, therefore,

$$|Q_n(z,\tau_n)|^{1/n} \leqslant \|Q_n\|_\infty^{1/n}$$

and

$$\liminf_{n\to\infty} |Q_n(z,\tau_n)|^{1/n} = \lim_{n\to\infty} |Q_n(z,\tau_n)|^{1/n} = 1 \leqslant \limsup_{n\to\infty} \|Q_n\|_\infty^{1/n}. \tag{4.13}$$

Since α is compactly contained in $\mathbb{D}$ and hence $\hat{\alpha}$ bounded away from $\mathbb{T}$, it follows that $Q_n(z,\tau_n) \in \mathscr{L}_n$, having poles in $\hat{\alpha}$ will be analytic in a disk of radius $\rho > 1$ for any $n = 1, 2, \ldots$. Hence, we have

$$\|Q_n\|_\infty = \max_{z\in\mathbb{T}\cup\mathbb{D}} |Q_n(z,\tau_n)| \leqslant \max_{z\in T_\rho} |Q_n(z,\tau_n)|, \quad T_\rho = \{z \in \mathbb{C} : |z| = \rho\}.$$

On the other hand, by Theorem 4.8(2), we have for $z \in T_\rho \subset \mathbb{E}$

$$\lim_{n\to\infty} |Q_n(z,\tau_n)|^{1/n} = \exp\{\lambda(\hat{z})\} \equiv \gamma(z).$$

Thus for $\varepsilon > 0$, there exists some n_0 such that for all $n > n_0$

$$\big| |Q_n(z,\tau_n)|^{1/n} - \gamma(z) \big| < \varepsilon$$

or, equivalently,

$$\gamma(z) - \varepsilon < |Q_n(z,\tau_n)|^{1/n} < \varepsilon + \gamma(z).$$

Hence, for sufficiently large n

$$\left[\max_{z\in T_\rho} |Q_n(z,\tau_n)|\right]^{1/n} \leqslant \max_{z\in T_\rho} |Q_n(z,\tau_n)|^{1/n} \leqslant \max_{z\in T_\rho}\{\varepsilon + \gamma(z)\}.$$

Let $\tilde{z}$ be a point in T_ρ where $\gamma(z)$ reaches its maximum. Then

$$\|Q_n\|_\infty^{1/n} \leqslant \left[\max_{z\in T_\rho} |Q_n(z,\tau_n)|\right]^{1/n} \leqslant \varepsilon + \gamma(\tilde{z}). \tag{4.14}$$

From its definition (4.10), it follows that for any $z = \rho e^{i\theta}$, $\lim_{\rho\to 1^+} \lambda(\hat{z}) = 0$ and hence $\lim_{\rho\to 1^+} \gamma(z) = 1$. Thus by taking $\tilde{z} = \rho e^{i\theta}$, we can write by (4.14)

$$\limsup_{n\to\infty} \|Q_n\|_\infty^{1/n} \leqslant 1. \tag{4.15}$$

Finally, by (4.13) and (4.15) the proof follows. □

5. Rates of convergence for MA, MPA and quadrature

In this section, we show that the results obtained in the last section for the asymptotics of the orthogonal and paraorthogonal functions can be used to obtain the rate of convergence for MA to $F_\mu(z)$.

Let us start with the MAs $R_n(z,\tau_n)$ of order (n,n) which were already shown to be given by (3.16). Note that the zeros of the denominator are the zeros of the paraorthogonal functions, which are distinct and lie on $\mathbb{T}$. Denote the approximation error as before by

$$E_n(z,\tau_n) = F_\mu(z) - R_n(z,\tau_n). \tag{5.1}$$

We are interested in obtaining

$$\limsup_{n\to\infty} |E_n(z,\tau_n)|^{1/n} \leqslant \gamma(z). \tag{5.2}$$

Clearly, the desirable property is that $\gamma(z)$ is less than one if possible for all $z \in \hat{\mathbb{C}} - \mathbb{T}$. We gave an expression for the error in Theorem 3.6 which involved the term δ_n of (3.23). We can now prove the following:

Lemma 5.1. *Let conditions* (S) *and* (C) *hold and assume* $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$ *Then for* δ_n *as defined in* (3.23) *we have*

$$\limsup_{n\to\infty} |\delta_n|^{1/n} \leqslant 1.$$

Proof. Since α is compactly contained in $\mathbb{D}$, there is some positive constant M such that $|\delta_n| \leqslant M\|Q_n\|_\infty$. Because $M^{1/n} \to 1$ and by Theorem 4.9, $\|Q_n\|_\infty^{1/n} \to 1$, the lemma is proved. □

The main result of this section is the following:

Theorem 5.2. *Assume conditions* (S) *and* (C) *hold and let* $\nu_n^\alpha \xrightarrow[n]{*} \nu^\alpha$. *Then we have the following estimates for the convergence rates for the MA* $R_n(z,\tau_n)$ *to the function* $F_\mu(z)$

(1) *For all* $z \in \mathbb{D}$: $\limsup_{n\to\infty} |E_n(z,\tau_n)|^{1/n} \leqslant \exp\{\lambda(z)\} < 1$;

(2) *For all* $z \in \mathbb{E}$: $\limsup_{n\to\infty} |E_n(z,\tau_n)|^{1/n} \leqslant \exp\{\lambda(\hat{z})\} < 1$, $\hat{z} = 1/\bar{z}$,

where $\lambda(z)$ *is as in* (4.10).

Proof. In this proof, we drop τ_n to simplify the notation. We use (3.22) to get

$$E_n(z) = \frac{2z\omega_{n-1}(z)\pi_{n-1}(z)}{\pi_n^2(z)Q_n^2(z)} \left[\int_{-\pi}^{\pi} \frac{\pi_n^2(t)Q_n^2(t)\,\mathrm{d}\mu(\theta)}{\pi_{n-1}(t)\omega_{n-1}(t)(t-z)} + \delta_n \right].$$

We consider first the case $z \in \mathbb{D}$. Then with $t = \mathrm{e}^{\mathrm{i}\theta}$

$$|E_n(z)| \leqslant 2|z| \left|\frac{\omega_{n-1}(z)}{\pi_n(z)}\right| \left|\frac{\pi_{n-1}(z)}{\pi_n(z)}\right| \frac{1}{|Q_n^2(z)|} \left[\int_{-\pi}^{\pi} \left|\frac{\pi_n(t)}{\pi_{n-1}(t)}\right| \left|\frac{\pi_n(t)}{\omega_{n-1}(t)}\right| \frac{|Q_n^2(t)|}{|t-z|}\,\mathrm{d}\mu(\theta) + |\delta_n| \right].$$

Thus, there exists a positive M (which does not depend on n) such that

$$|E_n(z)|^{1/n} \leqslant M^{1/n} \left|\frac{\omega_{n-1}(z)}{\pi_n(z)}\right|^{1/n} \left|\frac{\pi_{n-1}(z)}{\pi_n(z)}\right|^{1/n} \frac{1}{|Q_n^2(z)|^{1/n}} (S_n + |\delta_n|)^{1/n} \tag{5.3}$$

with

$$S_n = \|Q_n^2\|_\infty \max_{t\in\mathbb{T}} \left| \frac{\pi_n(t)}{\pi_{n-1}(t)} \frac{\pi_n(t)}{\omega_{n-1}(t)} \right| = \|Q_n^2\|_\infty \max_{t\in\mathbb{T}} |1 - \bar{\alpha}_n t|^2$$

because $|B_{n-1}(t)| = 1$ almost everywhere on $\mathbb{T}$. Because α is compactly contained in $\mathbb{D}$, $\lim_{n\to\infty} \max_{t\in\mathbb{T}} |1 - \bar{\alpha}_n t|^{2/n} = 1$ and we also know by Theorem 4.9 that $\limsup_{n\to\infty} \|Q_n\|_\infty^{1/n} = 1$. Thus

$\lim_{n\to\infty} S_n^{1/n} = 1$. We recall that also $\limsup_{n\to\infty} |\delta_n|^{1/n} \leqslant 1$ by Lemma 5.1. Note that for any sequence of positive numbers such that

$$\limsup_{n\to\infty} a_n^{1/n} = a \quad \text{and} \quad \limsup_{n\to\infty} b_n^{1/n} = b,$$

we have $\limsup_{n\to\infty}(a_n + b_n)^{1/n} = \max\{a, b\}$ and also $\lim_{n\to\infty} a_{n-1}^{1/n} = a$, so that it follows from (5.3) that

$$\limsup_{n\to\infty} |E_n(z)|^{1/n} \leqslant \limsup_{n\to\infty} \left|\frac{\omega_{n-1}(z)}{\pi_n(z)}\right|^{1/n} = \lim_{n\to\infty} \frac{|\omega_{n-1}(z)|^{1/n}}{|\pi_{n-1}(z)|^{1/n}}$$

$$= \lim_{n\to\infty} |B_{n-1}|^{1/n} = \exp\{\lambda(z)\}.$$

Since $\operatorname{supp}(v^\alpha)$ is compactly contained in $\mathbb{D}$, we have for $z \in \mathbb{D}$ and $x \in \operatorname{supp}(v^\alpha)$ that the Blaschke factor satisfies

$$\left|\frac{z - x}{1 - \bar{z}x}\right| < r(z) < 1.$$

Therefore, $\lambda(z) < 0$ and the proof is complete for $z \in \mathbb{D}$.

Next we consider $z \in \mathbb{E}$. From (5.3) we see that

$$\limsup_{n\to\infty} |E_n(z)|^{1/n} \leqslant \lim_{n\to\infty} \left\{ \left|\frac{\omega_{n-1}(z)}{\pi_n(z)}\right|^{1/n} \frac{1}{|Q_n^2(z)|^{1/n}} \right\}. \tag{5.4}$$

By Theorem 4.8,

$$\lim_{n\to\infty} |Q_n(z)|^{1/n} = \exp\{\lambda(z)\}, \quad z \in \mathbb{E}.$$

Using also Lemma 4.6, we see that (5.4) becomes

$$\limsup_{n\to\infty} |E_n(z)|^{1/n} \leqslant \limsup_{n\to\infty} |B_n(z)|^{1/n} \cdot \limsup_{n\to\infty} \frac{1}{|z - \alpha_n|^{1/n}} \cdot \exp\{-2\lambda(z)\}$$

$$= \exp\{-\lambda(z)\} = \exp\{\lambda(\hat{z})\} < 1.$$

We have used that for $z \in \mathbb{E}$ and the α_k complactly contained in $\mathbb{D}$ it should be clear that $\lim_{n\to\infty} |z - \alpha_n|^{1/n} = 1$. This completes the proof for $z \in \mathbb{E}$. □

As an example, we consider again the case where $\lim_{n\to\infty} \alpha_n = a \in \mathbb{D}$. Then $v^\alpha(t) = \delta_a$ and

$$\lambda(z) = \int \log\left|\frac{z - x}{1 - \bar{z}x}\right| \, \mathrm{d}\delta_a = \log\left|\frac{z - a}{1 - \bar{z}a}\right|.$$

Thus,

$$\limsup_{n\to\infty} |E_n(z)|^{1/n} \leqslant \left|\frac{z - a}{1 - \bar{z}a}\right| < 1, \quad z \in \mathbb{D},$$

and

$$\limsup_{n\to\infty} |E_n(z)|^{1/n} \leqslant \left|\frac{1 - \bar{z}a}{z - a}\right| < 1, \quad z \in \mathbb{E}.$$

Observe that when $\alpha_k = a$ for all $k = 1,2,\ldots$, i.e., in the case where only two points are used in the approximation, then the modified approximants are almost (one interpolation condition short) two-point Padé approximents. In that case we obtain the same result as in the case of a fixed limit. Thus if the α_k tend to a fixed point $a \in \mathbb{D}$, then from an asymptotic point of view, the MA for the multipoint case behave similar to the MA for the two-point case a and $1/\bar{a}$. If in particular $a = 0$, then the sequences of modified approximants for the two-point case, studied by Jones et al. [19] arise. It was proved in their paper that these converge locally uniformly in $\hat{\mathbb{C}} - \mathbb{T}$. We have added to their result that the convergence is geometric. See also [17]. Note also that, just as one would expect, the best rate is achieved for points z close to a and $1/\bar{a}$.

Recall that in Section 3 it was shown that the rational functions

$$R_n(z) = -\frac{\psi_n(z)}{\phi_n(z)} \quad \text{and} \quad R_n^{\times}(z) = \frac{\psi_{n*}(z)}{\phi_{n*}(z)} = \frac{\psi_n^*(z)}{\phi_n^*(z)}$$

represent MPAs for $F_\mu(z)$ of balanced order $(n, n+1)$ and $(n+1, n)$, respectively. $R_n(z)$ is analytic in $\mathbb{E}$ and $R_n^{\times}(z)$ is analytic in $\mathbb{D}$ for all $n = 1,2,\ldots$. In [3] it was proved that both sequences $\{R_n^{\times}(z)\}$ and $\{R_n(z)\}$ converge locally uniformly to $F_\mu(z)$. The first one in $\mathbb{D}$ and the second one in $\mathbb{E}$. Now we add to this the rate of convergence.

Theorem 5.3. *Assume conditions* (S) *and* (C) *hold and let* $\nu_n^{\alpha} \xrightarrow[n]{*} \nu^{\alpha}$. *Let*

$$E_n(z) = F_\mu(z) - R_n(z) = F_\mu(z) + \frac{\psi_n(z)}{\phi_n(z)} \quad \text{and} \quad E_n^{\times}(z) = F_\mu(z) - R_n^{\times}(z) = F_\mu(z) - \frac{\psi_n^*(z)}{\phi_n^*(z)}.$$

Then for z *in compact subsets of the indicated regions*:
(1) $\limsup_{n\to\infty} |E_n^{\times}(z)|^{1/n} \leqslant \exp\{\lambda(z)\}$, $z \in \mathbb{D}$,
(2) $\limsup_{n\to\infty} |E_n(z)|^{1/n} \leqslant \exp\{\lambda(\hat{z})\}$, $z \in \mathbb{E}$,
where $\lambda(z)$ *is as in* (4.10).

Proof. By using a substar conjugate, part (1) is immediately obtained from part (2). We thus only have to prove the second part. The latter follows from the error expression (3.24) using arguments which are practically the same as used in the proof of Theorem 5.2. □

Remark. In [22, Theorem 7] Pan formulates a similar result for an arbitrary measure. His proof is based on the fact that a certain expression $A_n(z)$ satisfies $A_n(z)^{1/n} \to 1$ locally uniformly. Therefore, it was shown that $A_n(z)$ is uniformly bounded from above, but we were not able to prove in our situation that it is uniformly bounded away from zero.

We should note however that it is possible to prove the previous theorem assuming only the conditions (B) and $\nu_n^{\alpha} \xrightarrow[n]{*} \nu^{\alpha}$. See [13, Theorem 9.10.2]. This however requires different techniques which are too remote from the thread of the current paper to include it here. A similar remark can be made about the next theorem.

To end this section, we give the rate of convergence for the R-Szegő quadrature formulas introduced in Section 3. We now prove

Theorem 5.4. *Let $\{\tau_n\}$ be a given sequence of complex numbers on $\mathbb{T}$. For each $n = 1,2,\ldots,$ let $\{x_{j,n}\}_{j=1}^n$ be the zeros of the paraorthogonal rational functions $Q_n(z,\tau_n)$ and consider the corresponding R-Szegő formulas $I_n\{f\}$ of* (3.11). *Then, under the same conditions as in the previous theorems it holds that*

$$\limsup_{n\to\infty} |I_\mu\{f\} - I_n\{f\}|^{1/n} \leqslant \rho < 1$$

for any analytic function f in a domain $G \in \mathscr{G}$, and where $\rho = \max\{\rho_1, \rho_2\}$ with

$$\rho_1 = \max_{z\in\Gamma\cap\mathbb{D}} \exp\{\lambda(z)\}, \quad \rho_2 = \max_{z\in\Gamma\cap\mathbb{E}} \exp\{\lambda(1/\bar{z})\},$$

$\lambda(z)$ as in (4.10) *and $\Gamma = \partial G$ the boundary of G.*

Proof. From (3.15), we can ensure that there exists a positive constant M such that

$$|I_\mu\{f\} - I_n\{f\}| \leqslant M\left(\max_{z\in\Gamma} |F_\mu(z) - F_n(z)|\right),$$

where $F_n(z)$ is the MA for F_μ of order (n,n) with poles $\{x_{j,n}\}_{j=1}^n$. Thus for all $z \in \Gamma \cap \mathbb{D}$, we have by Theorem 5.2 (1)

$$\begin{aligned}\limsup_{n\to\infty} |I_\mu\{f\} - I_n\{f\}|^{1/n} &\leqslant \limsup_{n\to\infty} \left[\max_{z\in\Gamma\cap\mathbb{D}} |F_\mu(z) - F_n(z)|^{1/n}\right] \\ &\leqslant \max_{z\in\Gamma\cap\mathbb{D}} \limsup_{n\to\infty} |F_\mu(z) - F_n(z)|^{1/n} \\ &= \max_{z\in\Gamma\cap\mathbb{D}} \exp\{\lambda(z)\} = \rho_1,\end{aligned}$$

where $\lambda(z)$ is given by (4.10). If $z \in \Gamma \cap \mathbb{E}$, one can use the second part of Theorem 5.2 to deduce in a similar way that

$$\limsup_{n\to\infty} |I_\mu\{f\} - I_n\{f\}|^{1/n} \leqslant \max_{z\in\Gamma\cap\mathbb{E}} \exp\{\lambda(1/\bar{z})\} = \rho_2.$$

The proof now follows. □

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